A Two-layer Semi Empirical Model of Nonlinear Bending of the Cantilevered Beam

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Abstract. This paper suggests a semi empirical model of nonlinear bending of the cantilevered beam. The model learns by using the results of measurement of nonlinear bending of the cantilevered metal pipe loaded by sinkers on the end. These studies are of interest for long-term forecasting of the behavior of building beams, various structural elements of load-lifting machines and mechanisms.

1. Introduction
Modeling complex technical objects is often hampered by insufficient knowledge of the processes occurring in them. The model of such an object in the form of a differential equation and boundary conditions can cause false ideas about the accuracy of modeling, since the structure and coefficients of the equations are not known accurately, even less accurately the boundary conditions are usually known. All these parameters are usually changing during the operation of the object, which makes the prediction of its state on the basis of the original model even more problematic.

To identify the state of an object, one can involve the results of observations of it, but the problem of identifying equations and boundary conditions from these data (the inverse problem) is usually much more complicated than the direct problem of solving a differential equation with boundary conditions. Previously, we solved such problems using our methodology for constructing the neural network model of the object by differential equations and additional data [1–8]. However, the training of neural networks requires a fairly large computational cost. To solve this problem, a new class of multi-layer models [9] was developed, with the help of which it is possible to do it without a complicated training procedure. This approach creates additional opportunities for combining classical and new methods. The essence of the approach is to apply known numerical methods for integrating differential equations to an interval with a variable upper limit. As a result, an approximate solution is obtained not in the form of a set of numerical values, but as a function of this upper limit. The method [9] can be used together with neural networks, but in this paper we managed without using them.

In this paper, this approach is tested on the bend of a cantilevered metal tube. The developed methods can be applied for long-term forecasting of the behavior of building beams, various structural elements of load-lifting machines and mechanisms, taking into account the real picture of wear, aging and corrosion of metal. In [10] the necessity of an estimation of influence of deterioration of the

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building equipment and the hand tool on working conditions of workers of building industry as the most traumatic kind of activity is revealed and justified. From this point of view, solving inverse problems in modeling the state of loaded elements will increase the reliability of forecasting results.

2. Material and methods
The measurements were carried out at the next experimental setup. A straight metal tube 1060 mm long was taken. And a mass of 116 g of circular cross-section with an outer diameter of 1 cm and a wall thickness of 2 mm, onto which labels were applied after 50 mm. One end of the tube, at a distance of 60 mm from the left end, was clamped in the vice, and loads with a weight of 100 g to 1300 g were attached alternately to the right with the help of a thread. A screen with a millimeter grid was located behind the experimental setup at a distance of 100 mm in a vertical plane. The horizontal and vertical lines of the grid were controlled by a bubble level with an error of no more than 3 mm per 1000 mm in length. Before the experimental installation at a distance of 1500 mm there was a camera, connected with a computer. In the center of the image was the middle of a tube with no load. To convert the coordinates from the pixel space into the space of a millimeter grid with optical distortion compensation, two calibration functions were constructed from two variables over a sufficiently large number of points. The resulting measurement error should not exceed 5 mm.

As a mathematical model we use the equation of a large static deflection of a thin homogeneous physically linear elastic rod under the action of distributed q and concentrated p forces [11].

\[
\frac{d^2 \theta}{dz^2} = \frac{mgL^2}{D} \left( \frac{m_i}{m} + z \right) \cos \theta = 0
\]

where \(D\) and \(L\) are the constant flexural rigidity and length of the rod; \(\theta\) - is the angle of inclination of the tangent; \(z = 1 - s/L\) - \(s\) - the natural coordinate of the curved axis of the rod, measured from the seal, \(m\) - the mass of the rod, \(m_i\) - the mass of the load. In the experiment performed, the distributed and concentrated forces were the weights of the rod and the load at the end.

The boundary conditions have the form:

\[
\frac{d \theta}{dz} \bigg|_{z=0} = 0; \quad \theta \bigg|_{z=1} = \theta_0
\]

The angle \(\theta\) is related to the coordinates of the points on the rod by the equalities:

\[
\frac{dx}{ds} = \cos(\theta); \quad \frac{dy}{ds} = \sin(\theta);
\]

It should be pointed out that this equation describes the object in question inaccurately - the tube has a creep zone near the seal and is not a rod. This situation is typical for many applied problems, when processes in a real object are described by an approximate model in the form of differential equations, which is usually solved numerically. In addition, usually there are measurement data obtained during the process of observing the object. By standard methods, the model of an object that takes into account these data is difficult to construct. By our methods, an approximate model is constructed by the equation (in the problem under consideration, these are equations (1-2)), the parameters of which are refined from the measurement data.

We rewrite equation (1) in the form:

\[
\frac{d^2 \theta}{dz^2} = a \left( \mu + z \right) \cos \theta,
\]
where \( a = \frac{mgL^2}{D} \), \( \mu = \frac{m}{m} \).

As indicated earlier, equation (3) describes the process of bending a rod with a large error. This statement was confirmed by numerical experiments. To construct a more adequate model, we move from system (2)-(3) to its approximate parametric solution \( x(s, \theta_0, a) \) and \( y(s, \theta_0, a) \). The parameters \( \theta_0, a \) are found by the method of least squares by minimizing the expression:

\[
\sum_{i=1}^{N} (x(s_i, \theta_0, a) - x_i)^2 + (y(s_i, \theta_0, a) - y_i)^2
\]

(4)

Here \( N \) is the number of points at which measurements were taken, \( \{x_i, y_i\} \) - the measured coordinates of the points corresponding to the marks on the rod at a distance \( s_i \) from the embankment. To build more accurate models, we applied the approach developed in [9]. Its essence with respect to equation (3) is that the known formulas for the numerical solution of differential equations should be applied not to the interval \([0,1]\), but to an interval with a variable upper limit \([0, z]\). In this case, instead of a table of numbers, we get a function \( \theta(z, \theta_0, a) \), and the parameters of the problem \( \theta_0, a \) are among its arguments.

From \( \theta(z, \theta_0, a) \) we pass to the original Cartesian coordinates, integrating (2) according to the Simpson formula for a variable-length interval:

\[
x(s) = \frac{s}{6M} \left( 1 + \cos \left( \theta(1-s/l) \right) + 4 \sum_{i=1}^{M} \cos \left( \theta \left( \frac{1-s/l}{2M} (2i - 1) \right) \right) + 2 \sum_{i=1}^{M-1} \cos \left( \theta \left( \frac{1-s/l}{M} i \right) \right) \right)
\]

(5)

\[
y(s) = \frac{s}{6M} \left( \sin \left( \theta(1-s/l) \right) + 4 \sum_{i=1}^{M} \sin \left( \theta \left( \frac{1-s/l}{2M} (2i - 1) \right) \right) + 2 \sum_{i=1}^{M-1} \sin \left( \theta \left( \frac{1-s/l}{M} i \right) \right) \right)
\]

(6)

In numerical experiments, we used this formula for \( M = 10 \). As a result, we get the dependences \( x(s, \theta_0, a) \) and \( y(s, \theta_0, a) \). The parameters \( \theta_0, a \), as mentioned above, are calculated by minimizing expression (4).

We present the results of calculations for two variants of this approach. The first variant consists in applying to (3) the above-mentioned modification [9] of Störmer's method [12]. As a result, we obtain an approximate solution of the form:

\[
\theta_i(z) \cong \theta_0 + \frac{z^2}{4} a(\mu_i + z) \cos(\theta_0 + \frac{z^2}{16} a(\mu_i + z))
\]

(7)

where \( \theta_0 = \theta(0) \) is the angle of the rod at its end.

Substituting (7) into (5) instead of \( \theta \), we obtain Simpson's formulas for \( x_i(s, \theta_0, a) \) and \( y_i(s, \theta_0, a) \). Parameters \( \theta_0 \) and \( a \) are not formally constrained. In fact, the accuracy of this solution is higher, the smaller the parameter \( a \), but in connection with the approximate character of equation (3) we are interested not in the smallness of the error of the solution of this equation, but in the accuracy of the correspondence to the measurement data.
The second variant consists in applying for the approximate solution of equation (3) an implicit Euler method [12] with two steps instead of the Störmer method. As a result, we obtain a system of two equations:

\[
\begin{align*}
\dot{\theta}(z) & \equiv \theta_0 + \frac{z^2}{4} a(\mu_i + z)\cos(\hat{\theta}(z)), \\
\theta(z) & \equiv 2\dot{\theta}(z) - \theta_0 + \frac{z^2}{4} a(\mu_i + z)\cos(\theta(z))
\end{align*}
\]

Whence we obtain an approximate solution:

\[
\theta_2(z) \equiv 2\frac{2\dot{\theta}_2(z) - \theta_0 + 0.25z^2a(\mu_i + z)}{\sqrt{1 + 0.5z^2a(\mu_i + z)(2\dot{\theta}_2(z) - \theta_0 + 0.25z^2a(\mu_i + z))} + 1}
\]

where:

\[
\dot{\theta}_2(z) = 2\frac{\theta_0 + 0.25z^2a(\mu_i + z)}{\sqrt{1 + 0.5z^2a(\mu_i + z)(\theta_0 + 0.25z^2a(\mu_i + z))} + 1}
\]

Substituting (8) into (5) instead of \( \theta \), we obtain Simpson's formulas for \( x_2(s, \theta_0, a) \) and \( y_2(s, \theta_0, a) \).

3. Calculation

Let's give the results of calculations for three values of the mass of the cargo \( m_2 = 300 \text{ g} \) and \( m_3 = 700 \text{ g} \). Figure 1 compares the measurement data and the results of calculations using formulas (7)-(8) for \( m_1 = 0 \text{ g} \). For the standard deviation of the measurement results from the theoretical curve is 1.71, the ratio of the residual variance to the variance of the sample is 0.0033. For the standard deviation of the measurement results from the theoretical curve is 0.22, the ratio of the residual variance to the sample variance is 0.0037.

![Figure 1](image)
The standard deviation of the measurement results from the theoretical curve \( x_1(s, \theta_0, a) \) is 1.71, the ratio of the residual variance to the variance of the sample is 0.0033. For the standard deviation of the measurement results from the theoretical curve \( y_1(s, \theta_0, a) \) is 0.22, the ratio of the residual variance to the sample variance is 0.0037.

The standard deviation of the measurement results from the theoretical curve \( x_2(s, \theta_0, a) \) is 1.72, the ratio of the residual variance to the variance of the sample is 0.0033. The standard deviation of the measurement results from the theoretical curve \( y_2(s, \theta_0, a) \) is 0.27, the ratio of the residual variance to the variance of the sample is 0.0056.

The discrepancy for the variable \( y \) is within the measurement error. The discrepancy with respect to the variable \( x \) is greater than the measurement error, which may indicate both the error of the model and the systematic error in the initial data. In the problem under consideration, we are interested in the shape of the rod bending, to which the measurement errors with respect to the variable \( x \) do not influence as much as the errors in the variable \( y \).

Figure 2 shows the discrepancies between the theoretical and experimental data for the variable \( y \), depending on the number of the measurement point, starting from the end of the rod.

Figure 2. Theoretical and experimental rod deflection curves obtained using formula 7 (the left figure (a)) and formula 8 (the right figure (b)). Here are values of the deflection on Cartesian coordinates \( Y \) and \( X \) in sm.

This figure shows that the deviations are not accidental. In figure 2a the error is maximal at the right end of the rod and decreases with approach to the left end. In figure 2b the error is maximal in the middle of the rod and decreases to its edges. This indicates the possibility of refinement of the model. Some methods for this refinement are discussed in the Results and Discussion section. For this task, this refinement was not carried out, since the problem is model and serves only to demonstrate methods. A similar conclusion can be drawn from the error graphs in the remaining cases. The corresponding graphs are not given in the article due to the limited space.

Figure 3 compares the measurement data and computational results using formulas (7)-(8) for \( m_2 = 300 \) g. For \( x_1(s, \theta_0, a) \) the standard deviation of the measurement results from the theoretical curve is 1.20, the ratio of the residual variance to the sample variance is 0.0016. For \( y_1(s, \theta_0, a) \) the standard deviation of the measurement results from the theoretical curve is 0.29, the ratio of the residual variance to the sample variance is 0.046.

For \( x_2(s, \theta_0, a) \) the standard deviation of the measurement results from the theoretical curve is 1.13, the ratio of the residual variance to the variance of the sample is 0.0014. For \( y_2(s, \theta_0, a) \) the
standard deviation of the measurement results from the theoretical curve is 0.14, the ratio of the residual variance to the variance of the sample is 0.0105.

Figure 3. Theoretical (curve) and experimental (points) rod deflection obtained using the formula 7 (the left figure (a)) and formula 8 (the right figure (b)) for the mass of the cargo $m_2 = 300$ g. Here $Y$ and $X$ are Cartesian coordinates in $sm$.

Figure 4 compares the measurement data and calculation results by formulas (7)-(8) for $m_3 = 700$ g.

Figure 4. Theoretical (curve) and experimental (points) rod deflection obtained using formulas 7 (the left figure (a)) and formula 8 (the right formula (b)) for the mass of the cargo $m_3 = 700$ g. Here $Y$ and $X$ are Cartesian coordinates in sm.

The conclusions that can be made on the basis of the error data are the same as in the case of a rod without a load. We note that similar results were obtained for other values of the mass of the cargo.

For $x_1(s, \theta_0, a)$ the standard deviation of the measurement results from the theoretical curve is 1.28, the ratio of the residual variance to the variance of the sample is 0.0019. For $y_1(s, \theta_0, a)$ the standard deviation of the measurement results from the theoretical curve is 0.69, the ratio of the residual variance to the variance of the sample is 0.0089. For $x_2(s, \theta_0, a)$ the standard deviation of the measurement results from the theoretical curve is 1.29, the ratio of the residual variance to the sample variance is 0.0019. For $y_2(s, \theta_0, a)$ the standard deviation of the measurement results from the theoretical curve is 0.56, the ratio of the residual variance to the sample variance is 0.0058. We note that similar results were obtained for other values of the mass of the cargo.
4. Results and Discussion

A typical situation for practice is the situation when experiments conducted with a real object contradict the mathematical model in the form of differential equations obtained on the basis of application of general physical principles. Usually, in such a situation, one of two approaches is used.

The first approach is to try to refine the physical model of the object and obtain differential equations that reflect the processes occurring in it more accurately. A simpler case leads to refinement of model parameters, which is mathematically expressed in coefficients of inverse problems. To solve such problems, there are a number of approaches, one of which is the use of neural networks [4, 7]. A variant (to which the problem considered in this article belongs) is also possible, when no choice of parameters allows us to reflect experimental data with reasonable accuracy. In this case, it is necessary to change the structure of the model, which can lead to a sharp complication of differential equations and does not always end in success in a reasonable time.

The second approach consists in refusing to use differential equations. In this case, the model of the object is constructed empirically by interpolation from the experimental data. Interpolation uses the selected system of functions, for example polynomials or neural networks.

The methods developed here allow us to apply the third approach. It consists in obtaining approximate semiempirical formulas on the basis of an inaccurate differential model and measurement results, which may also have a large error. Known theorems on the error of numerical methods [12] allow us to assert that we can obtain an arbitrarily exact approximation to the solution of a differential equation by using a partition into a sufficiently large number of intervals (which leads to more complicated formulas). This complication is justified only if the model in the form of differential equations accurately describes the real object. It should be emphasized that the standard numerical methods [12] allow us to obtain only tables of numbers. Unlike them, our approach allows us to obtain formulas without the use of interpolation, which can be refined from the experimental data.

Such an approach is convenient in practically interesting problems for constructing semiempirical models in a situation where the accuracy of describing an object by differential equations is small or unknown.

References


